

Second-order three-neighbor number-conserving cellular automata

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Abstract. In this study, number-conserving rules in second-order binary three-neighbor cellular automata are investigated. A polynomial representation of the local rules and its general expression are presented. Using this representation, the number-conservation condition is transformed into a system of linear equations. Solving this system under binary constraints reveals that 74 of the 2^{64} possible rules satisfy the number-conservation property. Furthermore, these rules are classified into 30 equivalence classes under standard equivalence relations. In addition, a necessary and sufficient condition for a cellular automaton to be number-conserving is derived.

Keywords: Second-order CA · Number-conserving · Discrete flux form.

A cellular automaton (CA) is a discrete dynamical system that models complex phenomena using simple local rules. Among them, number-conserving CAs play a key role in modeling systems where quantities such as particles or vehicles are conserved. A CA is number-conserving if the total sum of cell states is conserved over time; a well-known example is the traffic flow model given by elementary CA rule 184. Hattori and Takesue [1] derive a necessary and sufficient condition for a one-dimensional binary CA to have general additive conserved quantities. Boccara and Fuk s [2] derive a necessary and sufficient condition for a multi-state CA to be number-conserving. A comprehensive survey of further results on number-conserving CAs can be found in [3]. In this study, number-conserving rules for second-order three-neighbor binary cellular automata (SOCAs) are investigated, in line with our forthcoming paper [4]. A well-known subclass of SOCAs is the elementary reversible CA. The main objective is to enumerate all such rules and to derive a necessary and sufficient condition for number conservation. Among the resulting rules, the slow-to-start traffic flow model [5] is recovered.

Let $u_i^t \in \{0, 1\}$ be the state of cell i at discrete time t . The local rule $f : \{0, 1\}^6 \rightarrow \{0, 1\}$ for the SOCA is defined as

$$u_i^{t+1} = f(u_{i-1}^{t-1}, u_i^{t-1}, u_{i+1}^{t-1}, u_{i-1}^t, u_i^t, u_{i+1}^t).$$

The total number of local rules is $2^{64} \approx 1.84 \times 10^{19}$. For simplicity, let us denote $X = u_{i-1}^{t-1}, Y = u_i^{t-1}, Z = u_{i+1}^{t-1}, x = u_{i-1}^t, y = u_i^t$ and $z = u_{i+1}^t$. Using the disjunctive normal form in Boolean algebra, the local rule f is expressed as the following polynomial:

$$\begin{aligned} f(X, Y, Z, x, y, z) = & a_{63}XYZxyz + a_{62}XYZxy(1-z) \\ & + \cdots + a_0(1-X)(1-Y)(1-Z)(1-x)(1-y)(1-z), \end{aligned}$$

where $a_i \in \{0, 1\}$, $i = 0, 1, \dots, 63$ are entries of the rule table. This representation allows us to transform the number-conservation condition into a system of linear equations with binary constraints. Solving the linear system yields 74 number-conserving rules for the SOCAs. By considering standard equivalence relations (spatial reflection, Boolean conjugation, and their compositions), the 74 rules are classified into 30 equivalence classes. Taking temporal symmetry into account, which is specific to second-order systems, 15 representative rules are identified. The number-conserving rules identified here exhibit a variety of dynamical behaviors, including the propagation of localized structures, the formation of fractal-like patterns, and traffic-like dynamics. The following theorem gives a necessary and sufficient condition for the SOCA to be number-conserving.

Theorem 1. *Let $J : \{0, 1\}^4 \rightarrow \mathbb{R}$. The SOCA is number-conserving if and only if the local rule f can be written in one of the following forms*

$$f(X, Y, Z, x, y, z) = y + J(X, Y, x, y) - J(Y, Z, y, z), \quad (1)$$

$$f(X, Y, Z, x, y, z) = Y + J(X, Y, x, y) - J(Y, Z, y, z). \quad (2)$$

Representations (1) and (2) can be interpreted as discrete analogues of flux forms in conservation laws. Theorem 1 is a natural extension of the first-order case [1] to second-order CAs.

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